the blade stiffness k_b^i. The following dimensionless parameters are introduced:

$$\bar{k}_d = k_d/k_b; \qquad \bar{k}_c = k_c/k_b; \qquad \bar{k}_b^i = k_b^i/k_b$$
$$\bar{m} = m_d/m_b; \qquad \bar{\omega} = \omega/\sqrt{k_b/m_b} \tag{4}$$

where k_b is the average value of k_b^i.

The distribution of free vibration natural frequencies is plotted in Fig. 3 against the number of nodal diameters in the corresponding mode of vibration, for $\bar{k}_d = 10$, $\bar{k}_c = 1$, and $\bar{m} = 10$. There are two frequency bands because our model has two degrees of freedom per bay. Note that even quite complicated bladed-disk models can be reduced to the model in Fig. 2 by fitting the actual natural frequency distribution to that shown in Fig. 3 and identifying the values of $\bar{k}_d$, $\bar{k}_c$, and $\bar{m}$ for best fit.

Recall that since γ is a function of frequency, the localization strength differs for each rotor mode. However, in an optimization procedure, it would be too cumbersome to place constraints on the localization factors for each mode. Any typical or "interesting" mode can be chosen for this purpose. For convenience, the mode corresponding to an interblade phase angle of $\pi/2$ (with $N/4$ nodal diameter) is suggested. Note that the frequency corresponding to this mode is the median natural frequency, and in that sense this mode may be considered typical. The localization factor of the system can then be written as[2]

$$\gamma = \begin{cases} \dfrac{S^2 s^2}{8} & \text{for small } S \\ \ell n(Ss\sqrt{3}) - 1 & \text{for large } S \end{cases} \tag{5}$$

where s is the standard deviation of mistuning in blade stiffness,

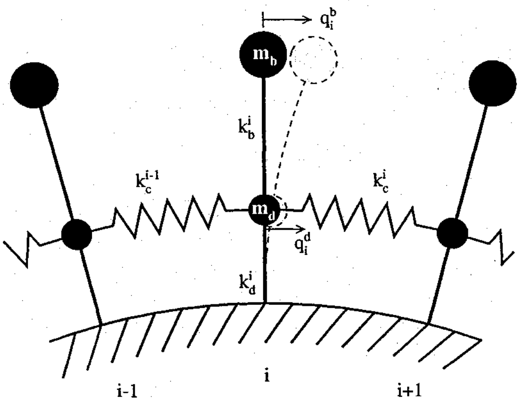

Fig. 2 The *i*th blade in an *N*-blade assembly with one blade coordinate and one disk coordinate per bay is shown with its two coupled adjacent blades.

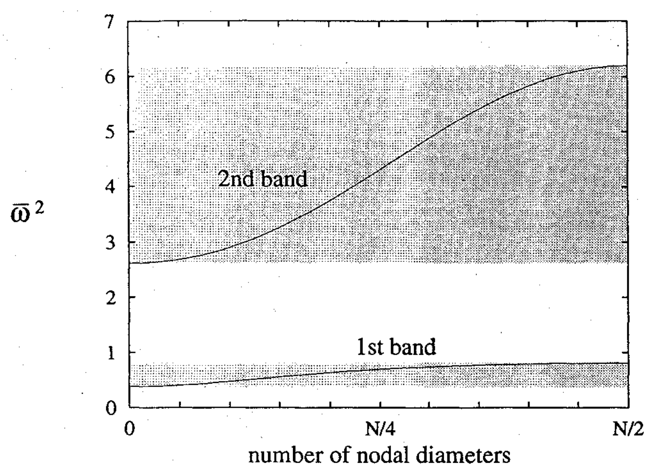

Fig. 3 Natural frequencies as a function of nodal diameter for the bladed-disk model in Fig. 2.

and S is a sensitivity measure defined by[2]

$$S = \frac{1}{4\bar{k}_c}\Big[2\bar{k}_c + \bar{k}_d - \bar{m} - 1 \mp \sqrt{(2\bar{k}_c + \bar{k}_d + \bar{m} + 1)^2 - 4\bar{m}(\bar{k}_d + 2\bar{k}_c)}\Big]^2 \tag{6}$$

where the minus sign refers to the first frequency band and the plus to the second. Note that Eq. (5) gives γ as a discontinuous function of the system parameters. This potential limitation in optimization applications could be overcome by redefining γ using a spline interpolation between the two segments of Eq. (5). Results from Monte Carlo simulations, if available, can be used to insure accurate interpolation.

The constraint on the influence of mistuning, Eq. (3), can now be expressed in terms of the tuned model parameters and the mistuning strength.

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Buckling of Polygonal and Circular Sandwich Plates

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Introduction

SANDWICH plates are extensively used in aerospace structures. When used in such structures, these plates are often subjected to in-plane forces. Thus, it is important to understand the buckling behavior of sandwich plates under in-plane forces.

Since the lighter core of the sandwich is usually relatively thick and low in stiffness when compared to isotropic plates, the effect of transverse shear deformation becomes significant and, thus, cannot be neglected in the buckling analysis. Otherwise, the buckling load will be overpredicted. A simple plate theory which allows for the effect of shear deformation was proposed by Reissner[1] for bending, and it was later extended to vibration and buckling problems by Mindlin[2] and Kollbrunner and Herrmann.[3] The theory assumes linear variations of the in-plane displacements through the thickness. Although the assumption gives rise to constant values of transverse shear stress distributions through the thickness (violating the condition of vanishing shear stresses at the surfaces), it has been found to produce reasonably accurate eigenvalue solutions by employing shear correction factors on the transverse shear modulii. This first-order shear deformation theory will be adopted for the buckling analysis of sandwich plates.

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This study presents the elastic buckling solutions for polygonal and circular sandwich plates under isotropic in-plane load in terms of their monolithic (Kirchhoff) counterparts. The considered polygonal plate is simply supported on all its straight edges whereas the circular plate may have either simply supported or clamped edges. The derived relationship linking the sandwich plate buckling solution and the Kirchhoff solution is exact. Therefore, exact sandwich plate solutions may be obtained whenever the Kirchhoff solutions are exact. Moreover, there are ample Kirchhoff buckling solutions in the open literature which one may draw from to compute the sandwich plate solutions.

Governing Buckling Equations of Sandwich Plates

Consider a sandwich plate whose core thickness is h_c and the thickness of the facings is h_f as shown in Fig. 1a. The Young's modulus of elasticity E, Poisson's ratio ν, and shear modulus G of the core and of the facings will be identified with subscripts c and f, respectively. The considered plate is of general polygonal shape or circular shape. The former plate shape has straight simply supported edges whereas the latter plate shape may have either simply supported or clamped edges. The plates are subjected to an isotropic in-plane load N, as shown in Figs. 1b and 1c.

On the basis of Reissner-Mindlin plate theory and assuming that the deformations are continuous through the sandwich plate thicknesses, the stress-displacement relations are given by[4]

$$M_{xx} = (D_C + D_f)\frac{\partial \psi_x}{\partial x} + (\nu_c D_C + \nu_f D_f)\frac{\partial \psi_y}{\partial y} \quad (1)$$

$$M_{yy} = (D_C + D_f)\frac{\partial \psi_y}{\partial y} + (\nu_c D_C + \nu_f D_f)\frac{\partial \psi_x}{\partial x} \quad (2)$$

$$M_{yx} = \tfrac{1}{2}((1-\nu_c)D_c + (1-\nu_f)D_f)\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right) \quad (3)$$

$$Q_x = \kappa^2(G_c h_c + 2G_f h_f)\left(\psi_x + \frac{\partial w_S}{\partial x}\right) \quad (4)$$

$$Q_y = \kappa^2(G_c h_c + 2G_f h_f)\left(\psi_y + \frac{\partial w_S}{\partial y}\right) \quad (5)$$

where M_{xx} and M_{yy} are the bending moments, M_{yx} is the twisting moment, Q_x and Q_y are the transverse shearing forces, all per unit length; w_S is the deflection of the middle plane of the sandwich plate; ψ_x and ψ_y are the rotations; κ^2 is the shear correction factor; D_c is the flexural rigidity of the core $= E_c h_c^3/[12(1-\nu_c^2)]$, and D_f is the flexural rigidity of the facings $= 2E_f h_f(3h_c^2/4 + 3h_c h_f/2 + h_f^2)/[3(1-\nu_f^2)]$.

The equilibrium equations of the plate are given by

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{yx}}{\partial y} = Q_x \quad (6)$$

$$\frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{yx}}{\partial x} = Q_y \quad (7)$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = N_S \nabla^2 w_S \quad (8)$$

By substituting Eqs. (1–5) into Eqs. (6–8), the three plate equations for buckling can be written as

$$\begin{aligned}\kappa^2(G_c h_c + 2G_f h_f)\left(\psi_x + \frac{\partial w_S}{\partial x}\right) &= (D_c + D_f)\frac{\partial^2 \psi_x}{\partial x^2} + (\nu_c D_c + \nu_f D_f)\frac{\partial^2 \psi_y}{\partial x \partial y} \\ &+ \tfrac{1}{2}((1-\nu_c)D_c + (1-\nu_f)D_f)\left(\frac{\partial^2 \psi_x}{\partial y^2} + \frac{\partial^2 \psi_y}{\partial x \partial y}\right) \quad (9)\end{aligned}$$

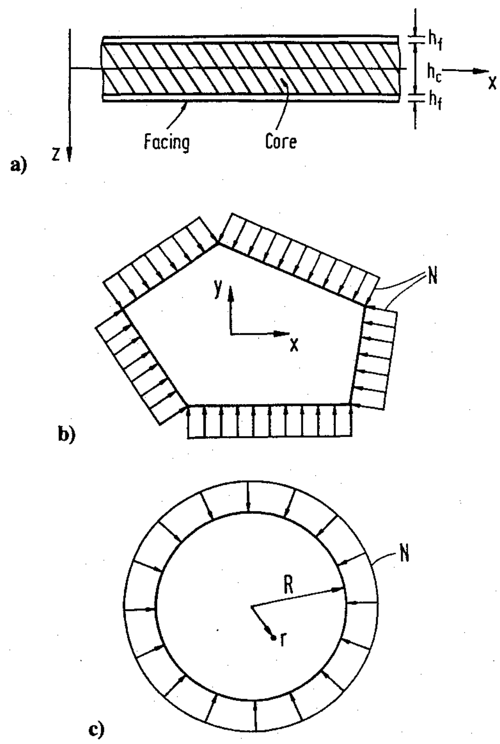

Fig. 1 Considered buckling problem of sandwich plates: a) cross section of sandwich plate, b) simply supported polygonal plate, and c) circular plate.

$$\begin{aligned}\kappa^2(G_c h_c + 2G_f h_f)\left(\psi_y + \frac{\partial w_S}{\partial y}\right) &= (D_c + D_f)\frac{\partial^2 \psi_y}{\partial y^2} + (\nu_c D_c + \nu_f D_f)\frac{\partial^2 \psi_x}{\partial x \partial y} \\ &+ \tfrac{1}{2}[(1-\nu_c)D_c + (1-\nu_f)D_f]\left(\frac{\partial^2 \psi_y}{\partial x^2} + \frac{\partial^2 \psi_x}{\partial x \partial y}\right) \quad (10)\end{aligned}$$

$$\kappa^2(G_c h_c + 2G_f h_f)\left(\Phi + \nabla^2 w_S\right) = N_S \nabla^2 w_S \quad (11)$$

where $\nabla^2(\bullet) = \partial^2(\bullet)/\partial x^2 + \partial^2(\bullet)/\partial y^2$ and the auxiliary function Φ is defined as

$$\Phi = \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} = \frac{M_{xx} + M_{yy}}{(1+\nu_c)D_c + (1+\nu_f)D_f} \quad (12)$$

By differentiating Eq. (9) and Eq. (10) with respect to x and y, respectively, then summing them up and using Eq. (11) leads to

$$\nabla^2\left(\nabla^2 w_S\right) + \frac{N_S}{(D_c + D_f)\left\{1 - \left[N_S/\kappa^2(G_c h_c + 2G_f h_f)\right]\right\}} \times \nabla^2 w_S = 0 \quad (13)$$

For the case of axisymmetric buckling of circular plates, the same Eq. (13) applies, but note that the Laplace operator is given by $\nabla^2(\bullet) = \mathrm{d}^2(\bullet)/\mathrm{d}r^2 + (1/r)\mathrm{d}(\bullet)/\mathrm{d}r$. For circular plates, Eq. (13) may be written as

$$\frac{\mathrm{d}^4 w_S}{\mathrm{d}\rho^4} + \frac{2}{\rho}\frac{\mathrm{d}^3 w_S}{\mathrm{d}\rho^3} + \left(1 - \frac{1}{\rho^2}\right)\frac{\mathrm{d}^2 w_S}{\mathrm{d}\rho^2} + \frac{1}{\rho}\left(1 + \frac{1}{\rho^2}\right)\frac{\mathrm{d}w_S}{\mathrm{d}\rho} = 0 \quad (14)$$

where

$$\lambda^2 = \frac{N_S}{(D_c + D_f)\left\{1 - \left[N_S/\kappa^2(G_c h_c + 2G_f h_f)\right]\right\}} \quad \text{and} \quad \rho = \lambda r \quad (15)$$

In Eq. (15), r is the radial coordinate. Note that Eq. (14) applies for both simply supported and clamped sandwich plates.

Relationship Between Sandwich and Kirchhoff Buckling Solutions

Simply Supported Polygonal Plate

For a Kirchhoff plate under an isotropic in-plane load, the governing fourth-order buckling equation is given by[5]

$$\nabla^2\left(\nabla^2 w_K\right) + \frac{N_K}{D}\nabla^2 w_K = 0 \tag{16}$$

where w_K is the deflection of the Kirchhoff plate and N_K the buckling load of the Kirchhoff plate. For simply supported polygonal Kirchhoff plate the deflection and the moment sum are zero at the boundary, i.e.,

$$w_K = 0 \qquad \text{and} \qquad \nabla^2 w_K = 0 \tag{17}$$

on the polygonal boundary. In view of Eq. (13) and Eq. (16) and the fact that the deflection $w_S = 0$ and the moment sum $\Phi = \nabla^2 w_S = 0$ on the boundary for the sandwich plates, one can readily deduce that

$$N_S = \frac{[(D_c + D_f)/D]N_K}{1 + \left(\{[(D_c + D_f)/D]N_K\}/\kappa^2(G_c h_c + 2G_f h_f)\right)} \tag{18}$$

Circular Plates

The governing equation for axisymmetric buckling of simply supported or clamped circular Kirchhoff plates under uniform radial compression is given by the same equation as in Eq. (14) (Ref. 6), except that

$$\lambda^2 = N_K/D \tag{19}$$

In view of Eq. (15) and Eq. (19), the buckling loads for circular sandwich plates and their Kirchhoff counterparts are also given by Eq. (18). The exact Kirchhoff buckling solutions are given by[7]

$$J_1(\lambda R) = 0 \Rightarrow N_K R^2/D = 14.6821, \tag{20a}$$

for clamped plates

$$\lambda R J_0(\lambda R) - (1 - \nu)J_1(\lambda R) = 0 \tag{20b}$$

for simply supported plates, where R is the radius of the plate and $J_0(\bullet)$ and $J_1(\bullet)$ are the zeroth-order and first-order Bessel functions, respectively.

Equation (18) furnishes an important exact relationship between the buckling load of the considered sandwich plates and the corresponding Kirchhoff plate solutions. Buckling loads of simply supported general polygonal sandwich plates and of simply supported/clamped circular sandwich plates may be readily computed from their Kirchhoff plate solutions. Thus, the necessity for a shear deformable plate buckling analysis is bypassed. Note that the buckling solutions of simply supported Kirchhoff plates of various shapes may be obtained from books and technical papers (such as Timoshenko and Gere,[7] Conway,[5] Wang et al.,[8] and Wang and Liew[9]).

Relation Between Sandwich Plate and Solid Mindlin Plate Solutions

The preceding derivation applies to the solid Mindlin plate solution as well. By letting the thickness of the facings, $h_f = 0$ and $h_c = h$ (i.e., $D_f = 0$, $D_c = D$), the sandwich plate becomes a solid Mindlin plate of thickness h. Thus, the key Eq. (18) reduces to

$$N_M = \frac{N_K}{1 + (N_K/\kappa^2 G h)} \tag{21}$$

where N_M is the buckling load of the solid Mindlin plate. This relationship was earlier derived by Irschik,[10] but it was expressed in terms of the alternative form of prestressed vibration solution instead of the Kirchhoff plate solution. Hong et al.[11] and Xiang et al.[12] gave the preceding relationship for circular Mindlin plates and rectangular Mindlin plates, respectively. The analogy between the buckling and vibration problem of polygonal Kirchhoff plates and the vibration problem of prestressed membrane is well known. Thus, one may use either of these solutions for Eq. (18).

In view of Eq. (18) and Eq. (21), the relationship between sandwich and solid Mindlin plate buckling solution is given by

$$\frac{[N_S/(D_c + D_f)]}{1 - \left[N_S/\kappa^2(G_c h_c + 2G_f h_f)\right]} = \frac{N_M/D}{1 - (N_M/\kappa^2 G h)} \tag{22}$$

If one has the solid Mindlin plate buckling solution for the considered plate shape, the sandwich plate solution may be obtained readily from Eq. (22) and vice versa.

Concluding Remarks

This study presents an exact relationship [Eq. (18)] between the buckling load of sandwich plates and Kirchhoff plates under isotropic in-plane load. The relationship applies for the simply supported general polygonal plate and simply supported and clamped circular plates. Exact buckling solutions for sandwich plates can be obtained from existing exact Kirchhoff solutions. Even vibration solutions of corresponding Kirchhoff plates or prestressed membranes can be used due to the analogies between these problems. The more complicated shear deformable buckling analysis for the considered sandwich plates may be bypassed because of the derived relationship.

This buckling relationship can also be used to check the convergence and accuracy of numerical shear deformable plate buckling solutions. Further, it can be used as a basic form for the development of approximate buckling formulas. Modification factors may be employed in such formulas to adjust the solutions for other plate shapes and boundary and loading conditions. Such approximate solutions and the application of the formula to simply supported plates of arbitrary curvilinear plan form, which may be approximated by a polygonal plate, will be discussed in a future paper.

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